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Letter to the Editor

Approximation closure method of FPK equations

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1. Introduction

In the analysis of non-linear random vibrations, the probability density function (PDF) of the system is the quantity of main concern. Because the exact solutions can be obtained only in very special cases for Fokker–Planck–Kolmogorov (FPK) equation or reduced FPK equation corresponding to non-linear systems with random excitations and the special cases can rarely be met in practice, approximate solution techniques are needed. Some approximation methods, e.g., the equivalent linearization method [1–2], the Gaussian closure method [3], the perturbation method [4], the Gram–Charlier series method [5–7], the equivalent non-linear system method [8,9], the stochastic average method [10,11] and the finite element method [12] have been developed.

Recently, Er [13] proposed an exponential polynomial function method, in which the PDF of the stationary responses of non-linear stochastic system is assumed to be an exponential function of polynomial in state variables with unknown parameters. Special measure is taken to satisfy FPK equation in the weak sense of integration with the assumed PDF. Evaluation of the parameters in the approximate PDF finally results in solving simultaneous quadratic algebraic equations. However, the simultaneous quadratic algebraic equations are difficult to solve. In this paper, a new arithmetic is developed to evaluate the parameters in the approximate PDF. Numerical calculation shows that the new arithmetic is suitable for strongly non-linear systems, and in some special cases even exact solutions of the FPK equations can be obtained.

2. A new arithmetic on the exponential polynomial closure method

The PDF of the response of non-linear stochastic system was approximated by $c\exp[Q_n(x;a)]$, where $Q_n(x;a)$ is an n-degree polynomial in state variables $x_1, x_2, \ldots, x_{n_x}$. Instead of solving FPK equation directly, the approximate PDF is substituted into FPK and a function in terms of x and

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unknown parameter a is factored out. After that, make the coefficients of the residual error vanishing, the unknown parameter a can be determined. To illustrate the new arithmetic, three examples are presented in this section.

Example 1. Consider the scalar diffusion process $X(t)$ satisfying the stochastic differential equation

$$\dot{X} = \tfrac{1}{2}(X - X^3 - \varepsilon X^5) + W(t), \tag{1}$$

where $W(t)$ is a Gaussian white noise with zero mean and correlation function $EW(t)W(t+\tau) = 2\pi S_0\delta(\tau)$, with S_0 being the spectral density of $W(t)$. $\delta(\tau)$ is a Dirac delta function and ε is a constant which represents the intensity of the non-linearity of the system.

For $S_0 = 1/\pi$, the stationary probability density function of the Markov process X is governed by the following stationary FPK equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(x - x^3 - \varepsilon x^5)p(x)\right] - \frac{\mathrm{d}^2 p(x)}{\mathrm{d}x^2} = 0. \tag{2}$$

The solution of Eq. (2) is

$$p(x) = C\exp\left(\frac{x^2}{4} - \frac{x^4}{8} - \frac{\varepsilon}{12}x^6\right), \tag{3}$$

where C is the normalization constant.

Now the exponential polynomial function method proposed by Er [13] is used to solve Eq. (2). It is assumed that Eq. (2) has the approximate solution of the form

$$p_n(x) = C\exp\left(\sum_{i=1}^{n} a_i x^i\right), \tag{4}$$

where C is the normalization constant, a_i, n are unknown parameters. Generally, the FPK Eq. (2) cannot be satisfied exactly with $p_n(x)$ because $p_n(x)$ is only an approximation of $p(x)$ and the number of unknown parameters is always limited in practice. Here, we present a new arithmetic to determine the unknown parameters a_i. Substituting $p_n(x)$ for $p(x)$ in Eq. (2) leads to the following residual error:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(x - x^3 - \varepsilon x^5)p_n(x)\right] - \frac{\mathrm{d}^2 p_n(x)}{\mathrm{d}x^2} = p_n(x)h_n(x),$$

where $h_n(x)$ is a polynomial function of the variable x. Because $p_n(x) \neq 0$, generally, the only possibility to satisfy Eq. (2) is $h_n(x) = 0$. However, usually $h_n(x) \neq 0$, because $p_n(x)$ is only an approximation of $p(x)$. In this case, we can make the coefficients $F_i(a_1, a_2, \ldots, a_n)$ of $x^i, i = 0, 1, \ldots, n-1$ in $h_n(x)$ vanish. This leads to

$$F_i(a_1, a_2, \ldots, a_n) = 0, \quad i = 0, 1, \ldots, n-1. \tag{5}$$

This means that $p_n(x)$ satisfies the reduced FPK Eq. (2) in the weak sense that the coefficients of low order power x^i in $h_n(x)$ vanish. Eq. (5) are n quadratic non-linear equations in terms of n undetermined parameters a_n. The algebraic equations can be solved with any available method to determine the unknown parameters.

For $\varepsilon = 0.05, n = 4$, four parameters a_1, a_2, a_3, a_4 are needed, four equations are formulated as follows:

$$\begin{gathered}0.5 - a_1^2 - 2a_2 = 0,\\ 0.5a_1 - 2a_1a_2 - 6a_3 = 0,\\ -1.5 + a_2 - 4a_2^2 - 6a_1a_3 - 12a_4 = 0,\\ (1.5 - 12a_2)a_3 - a_1(0.5 + 8a_4) = 0.\end{gathered} \tag{6}$$

The solution of Eq. (6) is $a_1 = 0, a_2 = \frac{1}{4}, a_3 = 0, a_4 = -\frac{1}{8}$.

For $n = 6$, six parameters $a_1, a_2, \ldots, a_6$ are needed, and can be solved as

$$a_1 = 0, \quad a_2 = \tfrac{1}{4}, \quad a_3 = 0, \quad a_4 = -\tfrac{1}{8}, \quad a_5 = 0, \quad a_6 = \tfrac{1}{240}.$$

In fact, $p_6(x)$ is the exact solution $p(x)$ of Eq. (2). Eq. (6) can be solved by elimination method, so it is easy to realize in computer program. However, the quadratic algebraic equations derived by Er [13]

$$\begin{gathered}-1.375 - a_1^2 - 6a_1a_3 - 4.75a_2 - 4a_2^2 - 48a_2a_4 - 27a_3^2 - 46.5a_4 - 240a_4^2 = 0,\\ a_1(-1.375 - 4a_2 - 24a_4) - 31.875a_3 - 36a_2a_3 - 360a_3a_4 = 0,\\ -5.875 - a_1^2 - 18a_1a_3 - 19.25a_2 - 12a_2^2 - 240a_2a_4 - 135a_3^2 - 310.5a_4 - 1680a_4^2 = 0,\\ -a_1(8.625 + 12a_2 + 120a_4) - 180a_2a_3 - 223.875a_3 - 2520a_3a_4 = 0,\end{gathered}$$

are more difficult to solve than Eq. (6).

Numerical results show that exact solution can be obtained with Eq. (5) for any value of ε in the case when $n = 6$; so, the new arithmetic is suitable for strongly non-linear systems. When $n = 8, 10$, the new arithmetic gives the same exact solution as in the case when $n = 6$.

For $\varepsilon = 0.05$, the exact and approximate PDFs of X are shown in Fig. 1. From Fig. 1, it is seen that the values of PDFs in the tails are very close to the exact solution.

Example 2. Consider the following oscillator with additive Gaussian white noise excitation:

$$\ddot{X} + \beta\dot{X} + X + X^3 + X^5 = W(t), \tag{7}$$

where $W(t)$ is a Gaussian white noise with zero mean and correlation function $EW(t)W(t+\tau) = 2\delta(\tau)$, the stationary probability density function $p(x_1, x_2)$ of the Markov vector $X = X_1$ and

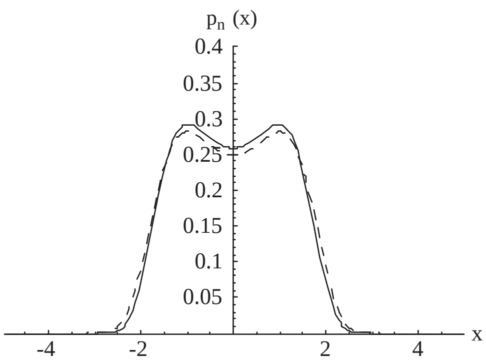

Fig. 1. The PDFs of X for Example 1. $n = 4$; exact $n = 6$.

$\dot{X} = X_2$ is governed by the following FPK equation:

$$-x_2 \frac{\partial p(x_1, x_2)}{\partial x_1} + \frac{\partial}{\partial x_2}[(\beta x_2 + x_1 + x_1^3 + x_1^5)p(x_1, x_2)] + \frac{\partial^2 p(x_1, x_2)}{\partial x_2^2} = 0. \tag{8}$$

The stationary PDF is obtained as [14]

$$p(x_1, x_2) = C \exp\left[-\beta\left(\frac{x_2^2}{2} + \frac{x_1^2}{2} + \frac{x_1^4}{4} + \frac{x_1^6}{6}\right)\right], \tag{9}$$

where C is the normalization constant.

Now the exponential polynomial function method is used to solve Eq. (8). It is assumed that Eq. (8) has the approximate solution of the form

$$\begin{aligned} p_n(x_1, x_2) = C \exp(&a_1 x_1 + a_2 x_2 + a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2 + \cdots \\ &+ a_{n_p} x_1^n + a_{n_p+1} x_1^{n-1} x_2 + \cdots + a_{n_p+n} x_2^n), \end{aligned} \tag{10}$$

where $n_p = n(n+1)/2$. Substituting $p_n(x_1, x_2)$ for $p(x_1, x_2)$ in Eq. (8) leads to the following residual error:

$$\begin{aligned} &- x_2 \frac{\partial p_n(x_1, x_2)}{\partial x_1} + \frac{\partial}{\partial x_2}[(\beta x_2 + x_1 + x_1^3 + x_1^5)p_n(x_1, x_2)] + \frac{\partial^2 p_n(x_1, x_2)}{\partial x_2^2} \\ &= p_n(x_1, x_2) h_n(x_1, x_2), \end{aligned}$$

where $h_n(x_1, x_2)$ is a polynomial function of the variables x_1, x_2. Making the coefficients of

$$x_1, x_2, x_1^2, x_1 x_2, x_2^2, \ldots, x_1^n, x_1^{n-1} x_2, \ldots, x_1 x_2^{n-1}$$

in $h_n(x_1, x_2)$ vanish leads to

$$F_i(a_1, a_2, \ldots, a_{n_p+n}) = 0, \quad i = 1, 2, \ldots, n_p + n. \tag{11}$$

Eq. (11) are $n_p + n$ quadratic non-linear equations in terms of $n_p + n$ undetermined parameters $a_1, a_2, \ldots, a_{n_p+n}$. For $\beta = 1, n = 4$, 14 parameters $a_1, a_2, \ldots, a_{14}$ are needed, and can also be solved as

$$a_1 = a_2 = a_4 = a_6 = a_7 = a_8 = a_9 = a_{11} = a_{12} = a_{13} = a_{14} = 0,$$
$$a_3 = a_5 = -0.5, \quad a_{10} = -1.25.$$

For $n = 6$, the approximate solution $p_6(x_1, x_2)$ is the exact solution $p(x_1, x_2)$. Numerical results show that exact solution can be obtained for any value of β in the case when $n = 6$, so the new arithmetic is suitable for a strongly damped system. When $n = 8, 10$, the new arithmetic gives the same exact solution as in the case when $n = 6$ (Fig. 2).

Numerical results show that the new arithmetic could give exact PDF solutions in the case when the PDF solution is of the type of an exponential polynomial, as shown in Examples 1 and 2.

Example 3. Consider another non-linear system with both additive and multiplicative random excitations, as shown by the following equation in Stratonovich's sense:

$$\ddot{X} + 2\alpha[1 + W_1(t)]\dot{X} + \omega^2[1 + W_2(t)]X + \beta_1\left(X^2 + \frac{\dot{X}^2}{\omega^2}\right)\dot{X} = W_3(t), \tag{12}$$

where $W_1(t), W_2(t), W_3(t)$ are independent Gaussian white noises in Stratonovich's sense with spectral densities k_1, k_2, k_3. For $\omega^2 k_2 = 4\alpha^2 k_1$, the stationary probability density function $p(x_1, x_2)$

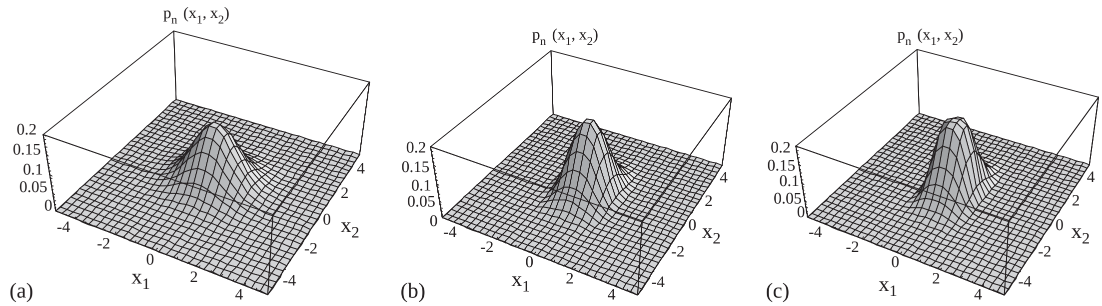

Fig. 2. The PDFs for Example 2. (a) $n = 2$; (b) $n = 4$; (c) exact. $n = 6$.

is obtained as [15]

$$p(x_1, x_2) = C\left(\varphi + x_1^2 + \frac{x_2^2}{\omega^2}\right)^{\gamma-\varphi\beta} \exp\left[-\beta\left(x_1^2 + \frac{x_2^2}{\omega^2}\right)\right], \tag{13}$$

where

$$\varphi = \frac{k_3}{k_2\omega^4}, \quad \beta = \frac{\beta_1}{k_2\omega^2}, \quad \gamma = \frac{2\alpha}{k_2\omega^2} + \frac{1}{2}.$$

Unlike the exact PDF solutions in Examples 1 and 2, this PDF solution is not of the type of exponential polynomial.

For $\alpha = 2, \beta_1 = 4, \omega = 1, k_1 = k_2 = k_3 = 2$, by using the new arithmetic as shown in Example 2, the approximate PDFs $p_n(x_1, x_2)$ can be obtained. The numerical results as shown in Fig. 3, showed that the approximate PDFs coincide with the exact one. When $n = 8, 10$, the new arithmetic gives similar results as in the case when $n = 6$.

3. Conclusions and discussions

Since the exact solution of the FPK equation is limited, approximate solution techniques are generally needed. Approximate closure technique is a method that assumes that the approximate PDF is a special kind of function, for example the exponential function of polynomial in state variable as shown in this paper. Other kinds of function, such as Lagrange polynomial function [16,17] may be used. In this paper, the approximate PDF is formed to be an exponential function of a polynomial in the state variables. The FPK equation is solved with a special technique such that the FPK equation is satisfied in the weak sense that the coefficients of the low order power of the state variables in residual error vanish. The present method is valid not only for single degree of freedom polynomial random systems, but also for multiple degree of freedom polynomial random systems. This method may be extended to general type of non-linearity, such as, for instance, sin x. One may expand sin x in Taylor series, then the new arithmetic can go on.

However, the choice of n depends on the numerical calculation experience. One may choose n in such a way, when different n gives similar results, such approximation result may be taken as exact one. Further efforts should be made for the problems such that if the approximate PDF p_n

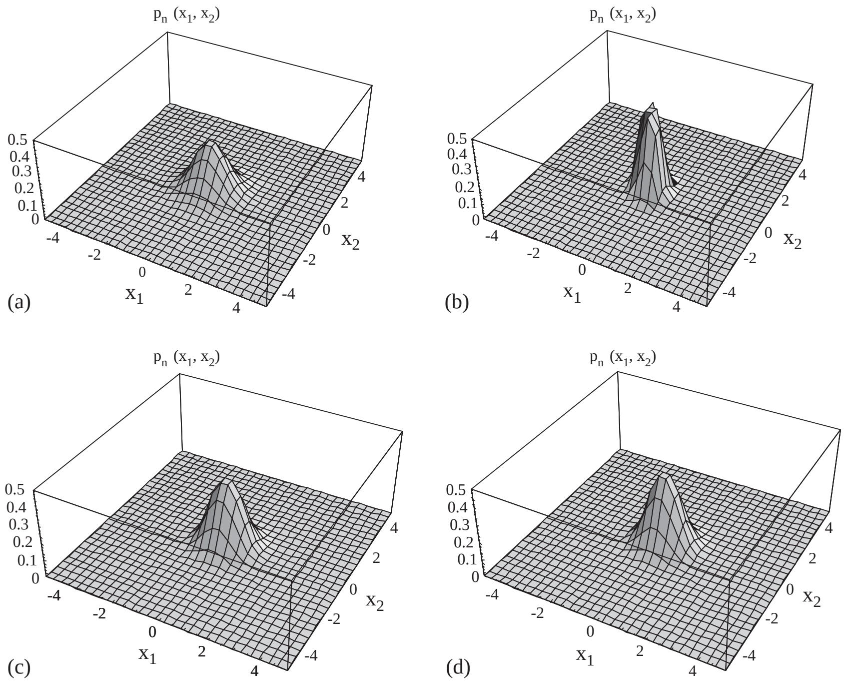

Fig. 3. The PDFs for Example 3. (a) $n = 2$; (b) $n = 4$; (c) $n = 6$; (d) exact.

converges to the exact PDF p as $n \to \infty$, or if gives n, how to determine the unknown parameters so that the approximate solution coincides with the exact one.

The new arithmetic proposed in this paper is an application of the exponential polynomial function method proposed by Er [13]. Evaluating the parameters using either the new approximation closure method or the exponential polynomial function method results in solving similar simultaneous quadratic algebraic equations. However, the quadratic algebraic equations derived by the new arithmetic are simpler than that by Er [13], and can be solved by the elimination method, so it is easy to realize in computer program.

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